

AN ALGORITHM FOR COMPUTING AN ELEMENT OF THE CLARKE GENERALIZED JACOBIAN OF A DIFFERENCE OF MAX-TYPE FUNCTIONS

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Abstract

We show that the algorithm for computing an element of the Clarke generalized Jacobian of a max-type function proposed by Zheng-da Huang and Guo-chun Ma in [8] can be extended to a much wider class of functions representable as a difference of max-type functions.

1 Introduction

Clarke's generalized differentiation constructions are employed in a variety of nonsmooth optimization techniques. The relevant generalized subdifferential and Jacobian are arguably the most common tools applied to a wealth of essentially nonsmooth problems (see the classical works [1, 4, 5, 11]). The new applications employing the Clarke Jacobian are still being developed; for example, a gradient bundle method essentially based on Clarke subgradients has recently proved successful in solving eigenvalue optimization problems [2, 3], and the nonsmooth Newton's Method [10] has been applied to important classes of nonsmooth problems, such as various nonlinear complementarity problems, stochastic optimization, semi-infinite programming, etc. For a brief but thorough overview of recent applications we refer the reader to [8].

Let a function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be locally Lipschitz around a point $x \in \mathbb{R}^n$. By $D_F \subset \mathbb{R}^n$ denote the set of points on which F is differentiable. For an $\bar{x} \in \mathbb{R}^n$ let

$$\partial F(\bar{x}) = \text{co} \limsup_{\substack{x \rightarrow \bar{x} \\ x \in D_F}} \{\nabla F(x)\}, \quad (1)$$

where by \limsup we denote the *outer set limit* (see [11]), i.e. the union of all limits of all converging subsequences, co is the convex hull, and $\nabla F(x)$ is the classical Jacobian of F at $x \in D_F$. The set $\partial F(\bar{x}) \subset \mathbb{R}^{n \times m}$ is the Clarke generalized Jacobian of F at \bar{x} . For a locally Lipschitz function this set is always nonempty and bounded (see [4]).

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Let a function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be such that each of its components is a pointwise maximum of a finite number of smooth functions, i.e., $F = (f_1, f_2, \dots, f_m)$,

$$f_i(x) = \max_{j \in J_i} f_{ij}(x), \quad \forall i \in I = \{1, \dots, m\}, \quad (2)$$

where $J_i, i \in I$ are finite index sets, and $f_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuously differentiable for all $x \in \mathbb{R}^n$.

It is impossible to compute the generalized Jacobian of max-type (or a difference of max-type) function from only the first-order information at hand (i.e. from the gradients of the component functions). The best one can do is to use the estimates like the bounds in [6] for quasidifferentiable functions. The algorithm suggested in [8] is essentially an elegant simplification of the method from [7]; both are motivated by the observation that for some important applications, such as Newton's method, the computation of the whole set is not required, we only need one arbitrary element that surely belongs to the generalized Jacobian.

Because of the max-type structure of the function, for every given point and any direction there exists an adjacent open set on which the max-type function is smooth, and hence the limit of the relevant Jacobians belongs to ∂F . The job of the algorithm is to carefully select the relevant gradients to build an element from the generalized Jacobian. In addition, the direction in [8] is chosen in a way to minimize the computation cost. The sole goal of this paper is to demonstrate that the original algorithm can be applied to a wider class of functions; we do not discuss the issues of finite precision and complexity here: this has already been addressed in [8] in detail.

We discuss the algorithm in Section 2 and prove its correctness in Section 3.

2 The Algorithm

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be such that $F = G - H$, where both G and H are max-type functions, i.e.

$$G = (g_1, \dots, g_m), \quad H = (h_1, \dots, h_m) \quad (3)$$

with

$$g_i(x) = \max_{j \in J_i} g_{ij}(x), \quad h_i(x) = \max_{k \in K_i} h_{ik}(x), \quad i \in I = \{1, \dots, m\}, \quad (4)$$

where $g_{ij}, h_{ik} : \mathbb{R}^n \rightarrow \mathbb{R}$ are C^1 functions, and J_i and K_i are finite index sets for all $i \in I$. We will also use the notation $F = (f_1, \dots, f_m)$ with $f_i = g_i - h_i, i \in I$.

For an $x \in \mathbb{R}^n$ and each $i \in I$ define the active index sets

$$J_i(x) = \left\{ j_0 \mid g_{ij_0}(x) = \max_{j \in J_i} g_{ij}(x) \right\} \text{ and } K_i(x) = \left\{ k_0 \mid h_{ik_0}(x) = \max_{k \in K_i} h_{ik}(x) \right\}.$$

By ∇f we denote the gradient of $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and by e_l we denote the l -th coordinate vector: $(e_l)_l = 1, (e_l)_j = 0, l \neq j$.

The Algorithm A1 is an extension of Algorithm 2.1 in [8]. The basic idea is to consider the individual subdifferentials of each of the functions $g_i, h_i, i \in I$ and from each one to choose one vertex (gradient) in a way that all the selected vertices correspond to the same direction in which all the aforementioned functions

are differentiable. Subroutine S1 does the selection per se, while Algorithm A1 on Step 1 iterates through the functions $g_i, h_i, i \in I$ and calls to Subroutine S1 on each iteration. On Step 2 of Algorithm A1 the gradients selected on the previous step are used to build an element $\xi \in \partial F(x)$.

Algorithm A1

Input: A point $x \in \mathbb{R}^n$, finite index sets J_i, K_i and functions $g_{ij}, j \in J_i$ and $h_{ik}, k \in K_i, i \in I$.
Step 1: For $i \in I$ compute $T_i(x) = \mathbf{S1}(x, J_i, \{g_{ij}\}_{j \in J_i})$, $S_i(x) = \mathbf{S1}(x, K_i, \{h_{ik}\}_{k \in K_i})$.
Step 2: Compute

$$\xi = (\nabla g_{1j_1}(x) + \nabla h_{1k_1}(x), \dots, \nabla g_{mj_m}(x) + \nabla h_{mk_m}(x))^T,$$

where $j_i \in T_i(x)$ and $k_i \in S_i(x)$ are chosen arbitrarily for each $i \in I$.

Subroutine S1

Input: A point $x \in \mathbb{R}^n$, a finite index set J and functions $g_j, j \in J$.
Step 1': Compute the active index set

$$J(x) = \{j_0 \in J \mid g_{j_0}(x) = \max_{j \in J} g_j(x)\},$$

let $T^0(x) = J(x)$.

Step 2': For $l = 1, \dots, n$ let

$$T^l(x) = \left\{ t_0 \in T^{l-1}(x) \mid \nabla g_{it_0}(x)^T e_l = \max_{t \in T^{l-1}(x)} \nabla g_{it}(x)^T e_l \right\}.$$

Output $T^n(x)$.

In the next section we prove the following result.

Theorem 1 *If $F = G - H$, where G and H are defined by equations (3)-(4), Algorithm A1 is well defined, and ξ generated by the algorithm is an element of $\partial F(x)$.*

3 Proof of the correctness of Algorithm A1

Our proof of Theorem 1 is essentially along the lines of the proof of Theorem 2.1 in [8], albeit is a bit shorter. We need to introduce a few definitions and technical results first.

Definition 2 *A continuous mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be PC^1 on an open set $U \subset \mathbb{R}^n$, if there exists a finite set of C^1 functions $f_j : U \rightarrow \mathbb{R}^m, j \in J$ (with $|J| < \infty$), such that for every $x \in U$, $f(x) = f_j(x)$ for at least one index $j \in J$.*

Recall that a directional derivative of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at a point $x \in \mathbb{R}^n$ in the direction y is the quantity

$$f'(x; y) = \lim_{t \downarrow 0} \frac{f(x + ty) - f(x)}{t}. \quad (5)$$

All PC^1 functions are directionally differentiable, i.e. the limit (5) exists for all directions $y \in \mathbb{R}^n$. The next result follows from the definition of the directional derivative. For a detailed discussion see [6, Chapter I, Corollary 3.2].

Lemma 3 Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a pointwise maximum of a finite number of smooth functions, i.e. for all $x \in \mathbb{R}^n$

$$f(x) = \max_{j \in J} f_j(x),$$

where $f_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j \in J$ are continuously differentiable, and J is a finite index set. Then for every $x \in \mathbb{R}^n$ the function f is directionally differentiable along an arbitrary direction $y \in \mathbb{R}^n$, and

$$f'(x; y) = \max_{j \in J(x)} f'_j(x; y) = \max_{j \in J(x)} \nabla f_j(x)^T y, \quad (6)$$

where $J(x)$ is the active index set:

$$J(x) = \{j_0 \in J \mid f_{j_0}(x) = \max_{j \in J} f_j(x)\}.$$

The following result is proved in [9, Lemma 2] (also see [8]).

Lemma 4 Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a PC^1 function in a neighborhood of $x \in \mathbb{R}^n$, then

$$\partial[F'(x; \cdot)](0) \subset \partial F(x),$$

where $[F'(x; \cdot)] : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the vector function of the directional derivatives of the components of F at the point x .

We are now in the position to prove our main result. The proof essentially follows the ideas of the proof of Theorem 2.1 in [8].

PROOF OF THEOREM 1 For each $i \in I$ let $T_i^0(x) := J_i(x)$, $S_i^0(x) := K_i(x)$, and for each $l \in \{1, \dots, n\}$ define the index subsets $T_i^l(x)$ and $S_i^l(x)$ recursively

$$T_i^l(x) = \left\{ t \in T_i^{l-1}(x) \mid \nabla g_{it}(x)^T e_l = \min_{j \in T_i^{l-1}(x)} \nabla g_{ij}(x)^T e_l \right\}; \quad (7)$$

$$S_i^l(x) = \left\{ s \in S_i^{l-1}(x) \mid \nabla h_{is}(x)^T e_l = \min_{k \in S_i^{l-1}(x)} \nabla h_{ik}(x)^T e_l \right\}. \quad (8)$$

It is not difficult to observe that the sets $T_i(x) = T_i^n(x)$ and $S_i(x) = S_i^n(x)$ are precisely the sets obtained after the execution of Step 1 of Algorithm A1. Since the index sets $J_i(x)$ and $K_i(x)$ are nonempty and finite, the minimal values of the scalar products in (7) and (8) are attained, and hence on every iteration of Step 2' of Subroutine S1 we generate nonempty sets. This means that after the execution of Step 1 of A1 we end up with nonempty finite sets $T_i^n(x)$ and $S_i^n(x)$, $i \in I$, so we can choose the corresponding indices $j_1, \dots, j_m, k_1, \dots, k_m$ on Step 2.

Let

$$\begin{aligned} \Gamma := & \{ \nabla g_{ij}(x) - \nabla g_{it}(x) \mid j \in J_i(x) \setminus T_i(x), t \in T_i(x), i \in I \} \\ & \cup \{ \nabla h_{ik}(x) - \nabla h_{is}(x) \mid k \in K_i(x) \setminus S_i(x), s \in S_i(x), i \in I \}. \end{aligned}$$

It follows from (7)-(8) that all the first nonzero components of all elements in Γ are positive.

Let ε denote a positive number smaller than the minimum value among the first nonzero components of all elements in Γ and let M be a positive number larger than the maximum value among the absolute values of all components of all elements in Γ , i.e. for each $\alpha = (0, \dots, 0, \alpha_k, \alpha_{k+1}, \dots, \alpha_n)^T \in \Gamma$ we have $|\alpha_i| < M$, $i = k, \dots, n$ and $\alpha_k > \varepsilon > 0$.

Let

$$\bar{y} = (-\lambda_1, -\lambda_2, \dots, -\lambda_n)^T,$$

where $\lambda_i > 0$, $\frac{\lambda_{i+1}}{\lambda_i} < \frac{\frac{\varepsilon}{M}}{1 + \frac{\varepsilon}{M}}$, $i = 1, \dots, n-1$. We have

$$\begin{aligned} \alpha^T \bar{y} &= -\sum_{i=k}^n \lambda_i \alpha_i \\ &\leq -\lambda_k \alpha_k + \sum_{i=k+1}^n \lambda_i |\alpha_i| \\ &\leq -\lambda_k \alpha_k + M \sum_{i=k+1}^n \lambda_i \\ &= -\lambda_k \alpha_k + M \lambda_k \sum_{i=k+1}^n \left(\frac{\frac{\varepsilon}{M}}{1 + \frac{\varepsilon}{M}} \right)^{i-k} \\ &< -\lambda_k \alpha_k + M \left(\frac{\varepsilon}{M} \right) \lambda_k \\ &= \lambda_k (\varepsilon - \alpha_k) \\ &< 0, \end{aligned}$$

which means that $\alpha^T \bar{y} < 0$ for all $\alpha \in \Gamma$. Observe that the set

$$U = \{y \mid \alpha^T y < 0; \forall \alpha \in \Gamma\}$$

is an open convex cone, which is nonempty since $\bar{y} \in U$. Hence, we have

$$(\nabla g_{ij}(x) - \nabla g_{it}(x))^T y < 0, \quad \forall j \in J_i(x) \setminus T_i(x), t \in T_i(x)$$

and

$$(\nabla h_{ik}(x) - \nabla h_{is}(x))^T y < 0, \quad \forall k \in K_i(x) \setminus S_i(x), s \in S_i(x),$$

which implies

$$\max_{j \in J_i(x)} \nabla g_{it}(x)^T y = \max_{t \in T_i(x)} \nabla g_{ij}(x)^T y \quad (9)$$

and

$$\max_{k \in K_i(x)} \nabla h_{is}(x)^T y = \max_{s \in S_i(x)} \nabla h_{ik}(x)^T y \quad (10)$$

for all $y \in U$. It follows directly from (7) and (8) that for every $i \in I$ we have

$$\nabla g_{it_1}(x) = \nabla g_{it_2}(x) \quad \forall t_1, t_2 \in T_i(x); \quad \nabla h_{is_1}(x) = \nabla h_{is_2}(x) \quad \forall s_1, s_2 \in S_i(x). \quad (11)$$

Hence, for each $i \in I$ we get

$$f'_i(x; y) = \nabla g_{ij}(x)^T y - \nabla h_{ik}(x)^T y = \xi_i^T y \quad (12)$$

for all $y \in U$, and any arbitrary choice of $j \in T_i(x)$ and $k \in S_i(x)$, i.e. $f'(x; y)$ is linear in y on U .

We next show that $\xi \in \partial_B F(x)$. Fix an arbitrary $y \in U$. Since U is an open cone, for all $t > 0$ we have $ty \in U$ together with a neighborhood; then from (12) we have $\nabla[F'(x; \cdot)](ty) = \xi$, and hence

$$\xi = \lim_{t \downarrow 0} \nabla[F'(x; \cdot)](ty) \in \partial[F'(x; \cdot)](0) \subset \partial F(x), \quad (13)$$

where the last inclusion follows from Lemma 4 and the observation that F is PC^1 . The proof is complete.

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